THE CALCULATION OF COMET EPHEMERIDES

BY JEREMY TATUM
Department of Physics, University of Victoria, Victoria, British Columbia

(*Received August 28, 1981; revised February 11, 1982*)

ABSTRACT

A brief description of how a basic comet ephemeris can be calculated from the orbital elements is given. In order to keep mathematics to a minimum, no proofs, derivations or detailed explanations are given.

RÉSUMÉ

On décrit une méthode pour calculer l'éphéméride d'une comète à partir des éléments orbitaux. Afin de retenir la mathématique au minimum, aucune épreuve, dérivation ou explication détaillée n'est donnée.

Introduction. In a recent article (Tatum 1982) I described how to calculate the position of a comet from measurements made on a photograph. There was one little step, however, that I missed out – namely, first find your comet!

Usually there is no difficulty, because ephemerides for periodic comets are published regularly in the *Handbook of the British Astronomical Association* and for new comets in the *Circulars* and the *Minor Planet Circulars* of the International Astronomical Union. Recently, however, at least partly because of a poor mail service, I experienced a long period when the I.A.U. Circulars, for one reason or another, were not reaching me, and this at a time when several comets were well placed for observation. This made me wonder whether it was possible for an ordinary mortal to calculate an ephemeris for himself.

The answer, I found, was a qualified yes.

A comet's orbit is described by certain *elements*, which are parameters that describe its shape, size and orientation. There are two stages to the calculation of an orbit and ephemeris. The first is to calculate the elements of the orbit from the available observations. The second is to calculate an *ephemeris* (that is, the day-to-day predictions of the right ascension and declination) from the elements.

The first stage was described by Newton as a *problema omnium longe difficillimum*. Now Newton was no ordinary mortal and I for one am prepared to take Sir Isaac's word for it. The problem of course has long been solved by such giants as Laplace and Gauss, but I prefer to leave such heroic calculations to their modern successors.

The calculation of a very precise ephemeris from the elements is no easy

problem either. The experts in this field include in their calculations such refinements as light-travel time, perturbations by the planets (which are sometimes very large) and non-gravitational effects caused presumably by the asymmetric evaporation of the material from the comet's nucleus.

If, however, one is satisfied with a simple Keplerian orbit without the inclusion of such complications, and would like to calculate an ephemeris that is usually adequate to find the comet in one's telescope, the calculation is almost trivial to one who possesses a good programmable calculator and knows how to use it. And even if the IAU *Circulars* are being safely delivered, it is still good fun to do it for oneself.

In this article I describe how to do the calculation. I unashamedly give no explanation of how to derive the various formulae, since this would result in much more mathematics, which would probably deter the very people for whom this article is intended. Almost nothing is original, and the material can be found in most standard textbooks on celestial mechanics, though I may possibly have added to the vast number of methods that have been proposed for the very rapid calculation of the true anomaly.

The Orbital Elements. An orbit can be an ellipse, a parabola or a hyperbola. Since a parabola represents a sort of dividing line between an ellipse and a hyperbola, and has an eccentricity of exactly unity, one might suppose that parabolic orbits never occur. However, most newly-discovered comets have very eccentric orbits and are observed over only a very short arc near perihelion. Very often it becomes impossible to distinguish between an ellipse and a parabola from such a short arc, and an attempt to calculate the major axis of an ellipse is not very reliable. Therefore in practice it is very usual to fit a parabola rather than an ellipse to the observations of a new comet seen over a short arc. Hyperbolic orbits are very rare. An orbit can be hyperbolic if the comet originated outside the solar system and is just passing through; or a previously elliptical orbit can be perturbed by Jupiter into a hyperbolic one, in which case the comet may be hurled out of the solar system. In practice it turns out that the twenty or so hyperbolic orbits that have been observed had eccentricities only very slightly in excess of unity, and probably all of these were orbits that had been changed by Jupiter from elliptic to hyperbolic. That is not to say there are no interstellar comets, but, if there are, they are so few and far between that we have not yet knowingly encountered one. In any case, hyperbolic orbits are rare, parabolic are common and there are a hundred or so elliptic orbits.

We describe the size and shape of an *elliptic* orbit by its semi-major axis a and its eccentricity e. In addition an elliptical orbit has a period P associated with it, related to a by $P^2 = a^3$ if P is expressed in sidereal years and a in astronomical units. All *parabolas*, of course, have the same shape ($e = 1$). There is no period,

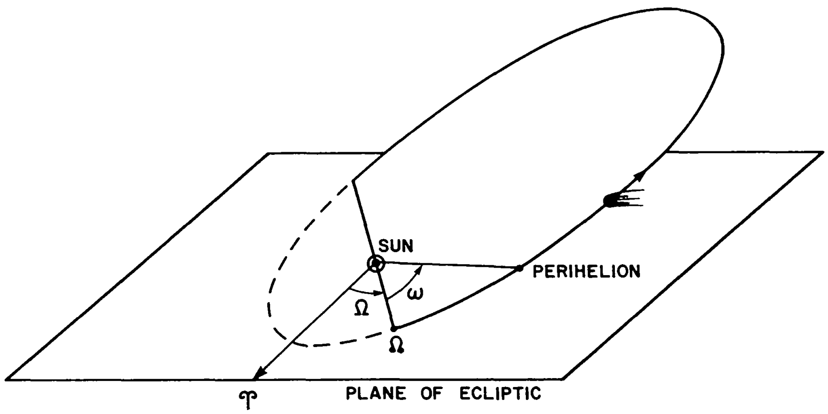

FIG. 1—Definition of Ω and ω. The perihelion point in this drawing does not seem to be the closest point to the Sun, but this is because the ellipse is being viewed obliquely.

neither is there a major axis, so the size is described by the perihelion distance q. The semi-transverse axis of a *hyperbola* is given the symbol a, and the eccentricity is e. There is of course no period. Sometimes the value of a is given as a negative number, but in this article I treat a always as a positive quality.

In addition to size and shape, we must describe the orientation of the orbit. Whether it is an ellipse, a parabola or a hyperbola, we do this by means of three angles ("Eulerian angles") as follows: i = inclination of the plane of the orbit to the plane of the ecliptic, Ω = ecliptic longitude of the ascending node, ω = argument of perihelion.

Figure 1 should help to make this clear. The ecliptic longitude of the ascending node is measured eastwards from the first point of Aries ♈ to the ascending node Ω. The argument of perihelion is measured from the ascending node in the plane of the orbit in the direction of the comet's motion. Both Ω and ω can have any value between 0° to 360°. The inclination can have any value from 0° to 180°. A little thought will show that negative values of i are not necessary. One can think of comets with inclinations greater than 90° as moving with retrograde motion. Other conventions for defining i, Ω and ω are possible, but the one described seems to be the most usual and, I think, the most sensible.

Auxiliary Quantities Independent of Time. Certain auxiliary quantities are useful in computing an ephemeris. These are denoted here by l, m, n, p, a', b, c, A, B, C, α', β, γ. These are functions of the elements i, Ω, ω and the obliquity of the ecliptic η, but are independent of time. They are useful in transforming coordinates from the plane of the orbit to equatorial coordinates. These should be calculated as the first stage in constructing an ephemeris, and are defined as follows:

$$l = \sin^2 \Omega + \cos^2 \Omega \cos^2 i \tag{1}$$

$$m = \sin^2 i \tag{2}$$

$$n = \sin 2\eta \cos \Omega \sin i \cos i \tag{3}$$

$$p = \cos \Omega \cos i \tag{4}$$

$$a' = (\cos^2 \Omega + \sin^2 \Omega \cos^2 i)^{1/2} \tag{5}$$

$$b = (l \cos^2 \eta + m \sin^2 \eta - n)^{1/2} \tag{6}$$

$$c = (l \sin^2 \eta + m \cos^2 \eta + n)^{1/2} \tag{7}$$

The positive square root is to be taken in equations (5) to (7). After this point, the quantities l, m, n are no longer required; they were needed only for the calculation of b and c.

$$A = \sin^{-1}\left(\frac{\cos \Omega}{a'}\right) = \cos^{-1}\left(-\frac{\sin \Omega \cos i}{a'}\right) \tag{8}$$

$$B = \sin^{-1}\left(\frac{\sin \Omega \cos \eta}{b}\right) = \cos^{-1}\left(\frac{p \cos \eta - \sin i \sin \eta}{b}\right) \tag{9}$$

$$C = \sin^{-1}\left(\frac{\sin \Omega \sin \eta}{c}\right) = \cos^{-1}\left(\frac{p \sin \eta + \sin i \cos \eta}{c}\right) \tag{10}$$

It is necessary to calculate A from both parts of equation (8), in order to determine the quadrant of A. Both $\sin^{-1}$ and $\cos^{-1}$ are double-valued, but only one value of A, the correct one, will be common to both. A, B and C can conveniently be expressed in degrees. After this point the quantities, i, Ω and p are no longer required; locations where they may have been stored in a calculator can now be freed for other variables.

Finally,

$$\alpha' = A + \omega \tag{11}$$

$$\beta = B + \omega \tag{12}$$

$$\gamma = C + \omega \tag{13}$$

and after this point A, B, C and ω (as well as l, m, n, p, i, Ω) are no longer required. However, a', b, c, α', β, γ, (which are not functions of time) will be needed later. I have used a prime on a' and α' so as to avoid confusion with the semi-major axis and right ascension respectively.

Usually we work with the equator and equinox of 1950.0 throughout. In that case, $\eta = 23^\circ.44812$.

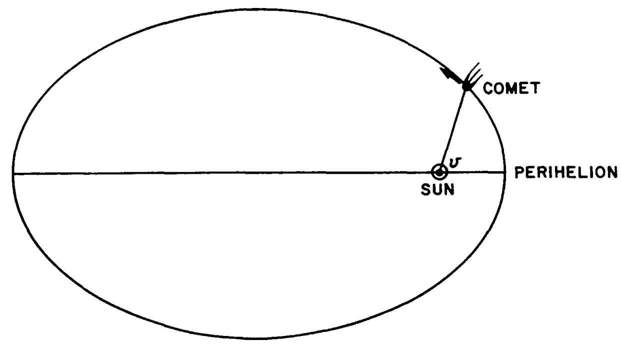

FIG. 2—Definition of v.

The True Anomaly. The next step in the calculation is to find the true anomaly v at time t, which is illustrated in figure 2. It is the angle subtended at the Sun between the comet at time t and the perihelion point of the comet's orbit. The comet passes perihelion at time T, so that the time interval between perihelion passage (true anomaly = 0) and true anomaly = v is $t - T$. The procedure for finding v at time t is different according to whether the orbit is an ellipse, a parabola or a hyperbola.

Ellipse.

First form a quantity M, known as the *mean anomaly*, from the equation

$$M = \frac{2\pi(t - T)}{P} \tag{14}$$

or from

$$M = \frac{2\pi(t - T)}{|a|^{3/2}} \tag{15}$$

In equation (14) $t - T$ and P must be expressed in the same units. In equation (15) $t - T$ must be expressed in sidereal years and a in astronomical units. The mean anomaly is actually an angle expressed in radians. It has a simple geometric interpretation, which need not, however, concern us here.

Now we calculate an angle E, the *eccentric anomaly*, from the equation

$$E = \frac{M + e(\sin E - E\cos E),}{1 - e\cos E} \tag{16}$$

which is a form of an equation commonly known as *Kepler's equation*. The eccentric anomaly also has a simple geometric interpretation, though it is not essential to describe it here. Some remarks on techniques for the solution of Kepler's equation are given later.

The true anomaly v is then calculated from

$$\cos v = \frac{\cos E - e}{1 - e \cos E} \tag{17}$$

Parabola.

First form a quantity M defined by

$$M = \frac{3\pi\sqrt{2}\,(t - T)}{q^{3/2}} = \frac{13.3286488\,(t - T)}{q^{3/2}} \tag{18}$$

where $t - T$ is in sidereal years and q is in astronomical units. Then calculate a quantity u from the equation

$$u = \frac{2u^3 + M}{3(1 + u^2)} \tag{19}$$

Finally calculate the true anomaly from

$$v = 2 \tan^{-1} u. \tag{20}$$

The sign of v is the same as the sign of M.

Hyperbola.

The true anomaly in a hyperbolic orbit can be elegantly expressed in terms of hyperbolic functions (sinh, cosh, etc.), but in this article I am not so concerned with mathematical elegance as with ease of understanding and with rapidity of calculation. Probably some readers are less familiar with hyperbolic functions than with the ordinary trigonometric functions (this is certainly true of myself) and few calculators have sinh or cosh buttons. The calculation described below may lack elegance, but it is very fast and well adapted to electronic hand calculators.

First form a quantity M from equation (15). Unlike the elliptical case I do not know of any particular geometrical significance of M though it is possible that there is one.

Then calculate a quantity u from

$$u = \frac{2u[e - u(1 - |M| - \ln u)]}{u(eu - 2) + e} \tag{21}$$

Finally, calculate the true anomaly from

$$\cos v = -\frac{u(u - 2e) + 1}{u(eu - 2) + e} \tag{22}$$

The sign of the true anomaly is the same as the sign of M. For a hyperbolic orbit the

largest possible value of v, approached as $t \to \infty$, is $\cos^{-1}(-e^{-1})$. M and u are then both infinite.

Remarks on the Solutions of the Equations. In this section I make some practical remarks concerning the solutions of the equations, especially equations (16), (19) and (21). It is suggested that, on first reading, this section be omitted. The reader can assume that he has succeeded in calculating the true anomaly v and can move to the section on heliocentric distance.

Two small points can be mentioned first. In equation (16) M and E are angles and should be expressed in radians. It is perhaps also worth mentioning that in equations (16), (17), (21) and (22) the quantity e is the eccentricity, not the base of natural logarithms.

It will doubtless be noticed that equations (16), (19) and (21) at first appear to be clumsy and inelegant; by algebraic manipulation they can be re-written in apparently much "simpler" forms. For example, equation (19) could be re-written as

$$u = M - \tfrac{1}{3} u^3. \tag{23}$$

Or again, a u will cancel on each side of equation (21).

The reader is urged, however great the temptation, not to do this. Considerable trouble has been taken to write these equations in a form that leads to the most rapid calculation on a programmable electronic hand-calculator. A recasting of the equations could increase the amount of calculation ten or even a hundred times.

In equation (16), E appears on the right-hand side. Likewise u appears on the right-hand side of equations (19) and (21). Therefore these equations have to be solved by iteration. Convergence is extremely rapid if the equations are not "simplified". To start the iteration, it helps to make a good "first guess". Graphs to supply a good first guess are given in figures 3, 4, and 5.

The Heliocentric Distance. The heliocentric distance, r, also called the length of the radius vector, is shown in figure 2, and is just the distance of the comet from the Sun. It is calculated from

$$\text{Ellipse:} \quad r = \frac{a(1 - e^2)}{1 + e \cos v} \tag{24}$$

$$\text{Parabola:} \quad r = \frac{2q}{1 + \cos v} \tag{25}$$

$$\text{Hyperbola:} \quad r = \frac{a(e^2 - 1)}{1 + e \cos v} \tag{26}$$

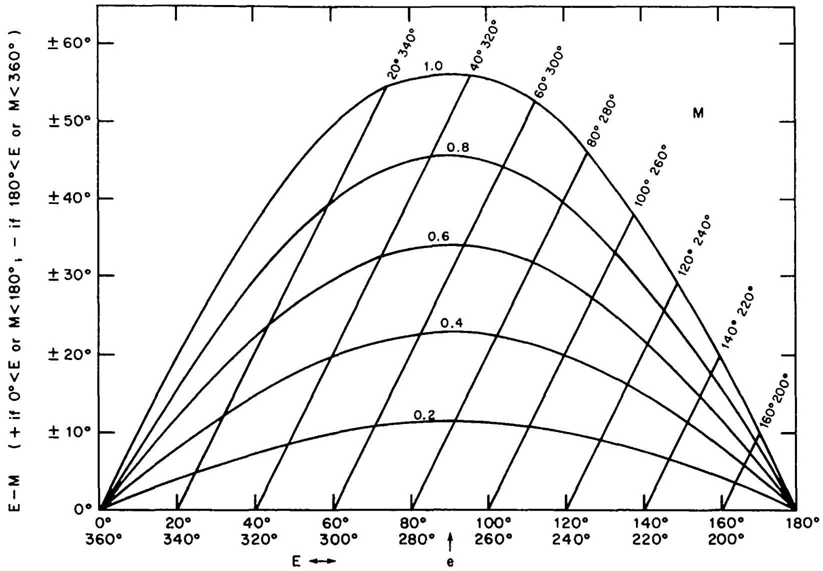

FIG. 3—Approximate solution of equation (16).

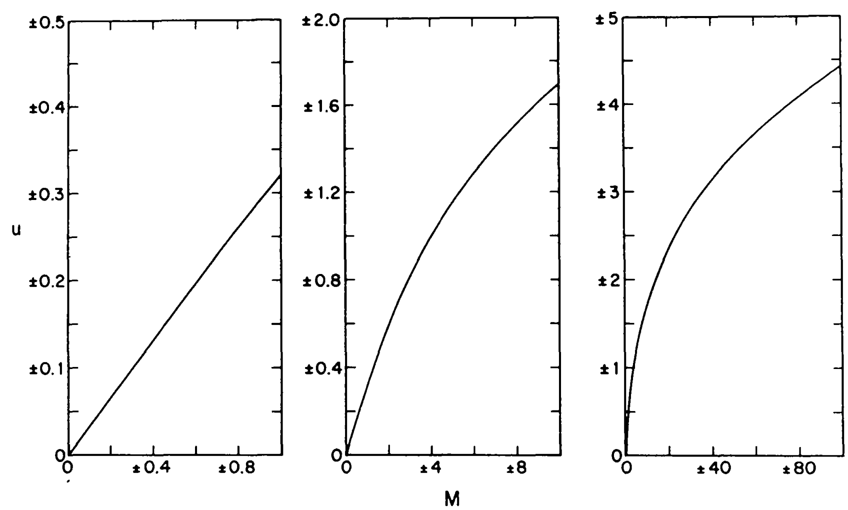

FIG. 4—Approximate solution of equation (19).

From this point onwards there is no difference in the calculation of an ephemeris between an elliptic, a parabolic or a hyperbolic orbit.

Geocentric Equatorial Coordinates. We next calculate the geocentric rectangular coordinates (x_c, y_c, z_c) of the comet. These are coordinates centred on the Earth, in which the xy-plane is the plane of the equator, with the x-axis directed towards the first point of Aries, the y-axis directed 90° east of this, and the z-axis towards the north pole. To do this we need to transform the coordinates (r, v), which are in

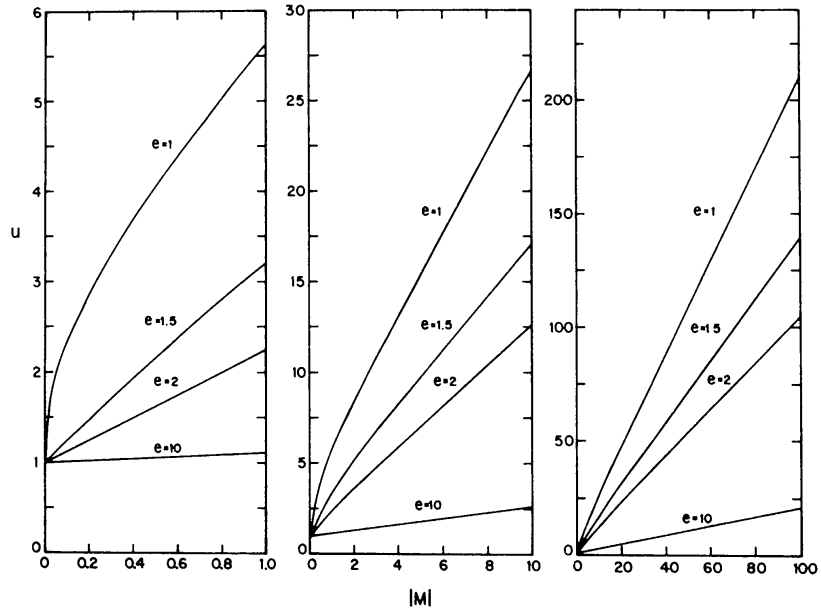

FIG. 5—Approximate solution of equation (21).

the plane of the orbit, first to the plane of the ecliptic using i, Ω, and ω, and then to the plane of the equator, using the obliquity η. We have already prepared for this part of the calculation by computing the auxiliary angles α', β, γ.

We also need to know the rectangular equatorial coordinates (x, y, z) of the sun, referred to the ecliptic and equinox of 1950.0. These are published for 0^hUT for every day of the year in Section C of the *Astronomical Almanac* (except for the 1981 edition), which was known before 1981 as the *American Ephemeris and Nautical Almanac*.

If we know (x, y, z) the calculation of (x_c, y_c, z_c) is simple:

$$x_c = r\,a' \sin(v + \alpha') + x \tag{27}$$

$$y_c = r\,b \sin(v + \beta) + y \tag{28}$$

$$z_c = r\,c \sin(v + \gamma) + z \tag{29}$$

Finally we calculate the right ascension α, declination δ and geocentric distance Δ from

$$\tan\alpha = y_c/x_c \tag{30}$$

$$\tan\delta = z_c/(x_c^2 + y_c^2)^{1/2} \tag{31}$$

$$\Delta = (x_c^2 + y_c^2 + z_c^2)^{1/2} \tag{32}$$

The ambiguity in the quadrant of α is determined by inspection of the signs of x_c and y_c, and the calculation is then complete.

Example. The elements of the elliptic orbit of Comet Grigg-Skjellerup 1980 II are, according to the 1982 *Handbook of the British Astronomical Association*, as follows:

$$T = 1982 \text{ May } 15.0023 \text{ E.T.}$$
$$a = 2.958981 \text{ A.U.}$$
$$e = 0.665683$$
$$\left.\begin{aligned} i &= 21^\circ\!.1366 \\ \Omega &= 212^\circ\!.6315 \\ \omega &= 359^\circ\!.3280 \end{aligned}\right\} 1950.0$$

Let us calculate its position for 1982 June 10.0 E.T. (E.T. = Ephemeris Time, which differs at present, 1982, from Universal Time by only about 53 seconds.) We recall that, referred to the equator and equinox of 1950.0,

$$\eta = 23^\circ\!.44812.$$

I shall proceed with the calculation using the same equation numbers as previously.

Auxiliary quantities

$$l = 0.907\,781\,296 \tag{1}$$
$$m = 0.130\,027\,093 \tag{2}$$
$$n = -0.206\,802\,170 \tag{3}$$
$$p = -0.785\,498\,653 \tag{4}$$
$$a' = 0.980\,913\,661 \tag{5}$$
$$b = 0.995\,708\,051 \tag{6}$$
$$c = 0.215\,345\,924 \tag{7}$$
$$A = 300^\circ\!.847 \tag{8}$$
$$B = 209^\circ\!.791 \tag{9}$$
$$C = 274^\circ\!.862 \tag{10}$$
$$\alpha' = 300^\circ\!.175 \tag{11}$$
$$\beta = 209^\circ\!.119 \tag{12}$$
$$\gamma = 274^\circ\!.190 \tag{13}$$

True Anomaly.

$$t - T = 25.9977 \text{ days.}$$

But $(t - T)$ in equation (15) must be expressed in sidereal years, and

$$1 \text{ sidereal year} = 365.25636 \text{ days.}$$

Therefore,

$$t - T = 0.071\,177 \text{ sidereal years,}$$

and therefore

$$M = 0.087\,862\,537 \text{ radians} \tag{15}$$

or

$$M \simeq 5°0.$$

Figure 3 indicates that a good first guess for the solution of equation (16) is

$$E \simeq 15° \simeq 0.26 \text{ radians}.$$

We put this value in the right hand side of equation (16) to obtain a better value of E, and we repeat this until the iteration converges. In this case it converges to 10 significant figures after two iterations:

$$\begin{aligned} E &= 14\overset{\circ}{.}735\,923\,48 = 0.257\,190\,383 \text{ radians} \\ E &= 14\overset{\circ}{.}735\,630\,81 = 0.257\,185\,275 \text{ radians}. \end{aligned} \tag{16}$$

Thus we find the true anomaly:

$$v = 32\overset{\circ}{.}199\,080\,76.$$

Heliocentric Distance.

$$r = 1.05402 \text{ A.U.} \tag{24}$$

Geocentric Equatorial Coordinates.

From page C22 of the 1982 *Astronomical Almanac* we find

$$x = +0.204\,8638$$

$$y = +0.912\,2643$$

$$z = +0.395\,5551.$$

From equations (27)–(32) we obtain, therefore,

$$x_c = -0.274\,5602 \tag{27}$$

$$y_c = -0.008\,4583 \tag{28}$$

$$z_c = +0.212\,8343 \tag{29}$$

$$\left.\begin{aligned} \alpha &= 12^{h}\,07\overset{m}{.}06 \quad (30) \\ \delta &= +37°46\overset{\prime}{.}1 \quad (31) \end{aligned}\right\} 1950.0$$

$$\Delta = 0.348 \text{ A.U.} \tag{32}$$

and the calculation is complete.

Jeremy B. Tatum,
Department of Physics,
University of Victoria,
Victoria, B.C.
V8W 2Y2

Reference

Tatum, J. B. 1982, *J. Roy. Astron. Soc. Canada* **76**, 97.